Let's do an example of finding a potential for a 3D vector field.

<u>Ex</u>: Determine whether the vector field

$$\vec{F} = \langle e^x \sin yz,\ ze^x \cos yz,\ ye^x \cos yz + 3z^2 \rangle$$

is conservative. If so, find a potential.

<u>Sol</u>: Let's just try to find one. If $\vec{F}$ is conservative

$$\vec{F} = \langle P, Q, R \rangle = \nabla f = \langle f_x, f_y, f_z \rangle$$

So,

$$f = \int P\,dx = e^x \sin yz + g(y,z)$$

$$\Rightarrow f_y = ze^x \cos yz + g_y(y,z) = Q = ze^x \cos yz$$

$$\Rightarrow g_y(y,z) = 0 \Rightarrow g(y,z) = h(z)$$

$$\Rightarrow f = e^x \sin yz + h(z)$$

$$\Rightarrow f_z = ye^x \cos yz + h'(z) = R = ye^x \cos yz + 3z^2$$

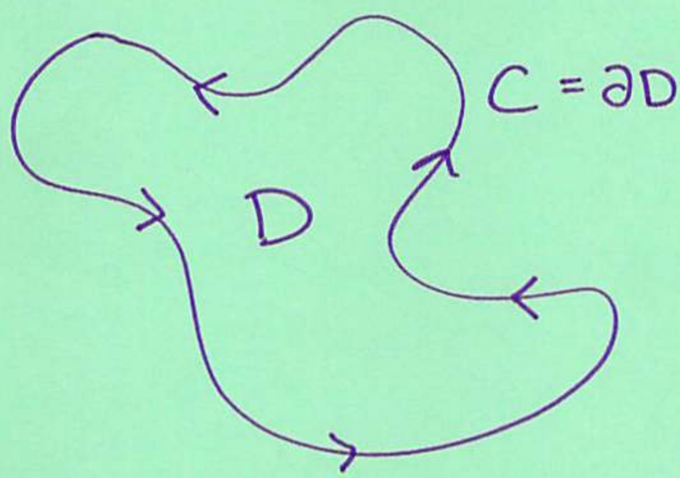
$$\Rightarrow h'(z) = 3z^2 \Rightarrow h(z) = z^3 + K$$

$\Rightarrow f = e^x \sin yz + z^3$ is a potential. $\diamond$

16.4 - Green's Theorem

Def: A curve C (in the plane) has *positive orientation* if it is traversed counterclockwise, exactly once. If C is the boundary of a region D in the plane, C has positive orientation if when you traverse C, D is on the left. We write $C = \partial D$ in this situation.

Ex:



Green's Theorem: Let C be a positively oriented, piecewise smooth, simple closed curve in the plane which bounds a region D. If P and Q have continuous first partials on a region containing D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Notation: For a closed curve C, we write

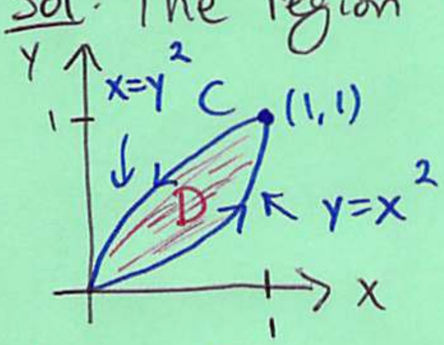
$$\oint_C P\,dx + Q\,dy$$

to imply that C has positive orientation. Recall that writing ∂D for the boundary of D implies ∂D has positive orientation. We could then rewrite Green's Theorem as:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{\partial D} P\,dx + Q\,dy.$$

Ex: Compute $\oint_C (y+e^{\sqrt{x}})\,dx + (2x+\cos y^2)\,dy$ where C is the boundary of the region bounded by $y=x^2$ and $x=y^2$.

Sol: The region is:



This is impossible to do directly, so, let's use Green's theorem on it:

$$\oint_C (y+e^{\sqrt{x}})\,dx + (2x+\cos y^2)\,dy = \iint_D [\underbrace{(2)}_{\frac{\partial Q}{\partial x}} - \underbrace{(1)}_{\frac{\partial P}{\partial y}}]\,dA = \iint_D dA$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} dx\,dy = \int_0^1 (\sqrt{y} - y^2)\,dy = \left(\frac{2}{3}y^{3/2} - \frac{1}{3}y^3\right)\Big|_0^1 = \frac{1}{3}$$

$\diamond$

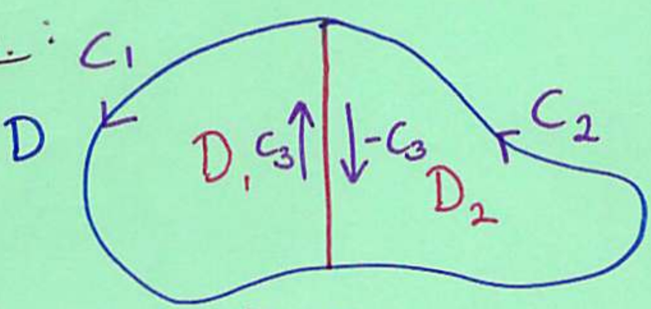
A common problem in double integration is finding the area of a region. We can use Green's theorem to translate this into a line integral around the boundary of the region:

$$\boxed{\text{Area}(D) = \iint_D 1\,dA = \oint_{\partial D} x\,dy = -\oint_{\partial D} y\,dx = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx}.$$

Sometimes we can come across regions with holes in them (e.g., annuli)... a fair question to ask would be if Green's theorem can deal with them.

Let's begin with the following: If $D = D_1 \cup D_2$, then $\iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA$.

E.g.:



$\partial D_1 = C_1 \cup C_3$

$\partial D_2 = C_2 \cup (-C_3)$

Then, Green's theorem tells us:

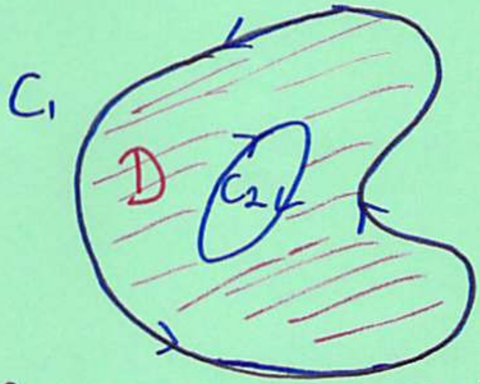
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D_1} (Q_x - P_y)\,dA + \iint_{D_2} (Q_x - P_y)\,dA$$

$$= \oint_{\partial D_1} P\,dx + Q\,dy + \oint_{\partial D_2} P\,dx + Q\,dy = \left(\int_{C_1} P\,dx + Q\,dy + \cancel{\int_{C_3} P\,dx + Q\,dy}\right)$$

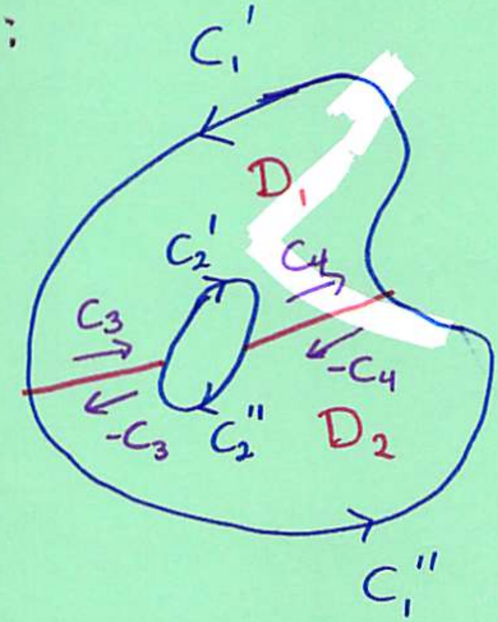
$$+ \left(\int_{C_2} P\,dx + Q\,dy - \cancel{\int_{C_3} P\,dx + Q\,dy}\right)$$

$$= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = \oint_{\partial D} P\,dx + Q\,dy$$

The purpose of this was to show that cutting regions doesn't change the integral. Now, if D has holes:



Notice that the orientation of ∂D gives C_1 the counterclockwise orientation and C_2 the clockwise orientation. Now, we can cut D into 2 regions without holes:



So, this gives:

$$\iint_D (Q_x - P_y)\,dA = \iint_{D_1} (Q_x - P_y)\,dA + \iint_{D_2} (Q_x - P_y)\,dA$$

$$= \left(\int_{C_1'} P\,dx + Q\,dy + \cancel{\int_{C_3} P\,dx + Q\,dy} + \int_{C_2'} P\,dx + Q\,dy + \cancel{\int_{C_4} P\,dx + Q\,dy}\right)$$

$$+ \left(\int_{C_1''} P\,dx + Q\,dy + \cancel{\int_{-C_4} P\,dx + Q\,dy} + \int_{C_2''} P\,dx + Q\,dy + \cancel{\int_{-C_3} P\,dx + Q\,dy}\right)$$

$$= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy$$

$$= \oint_{C_1} P\,dx + Q\,dy - \oint_{C_2} P\,dx + Q\,dy.$$

So, when there's holes, we integrate around the outside, the subtract off the integrals around the holes.

16.5 - Vector Reformulations of Green's Theorem

To do this, we include the plane into $\mathbb{R}^3$ as the xy-plane. This essentially means tacking on a 0 in the $\hat{k}$-component. So, $\vec{F}(x,y,z) = \langle P(x,y), Q(x,y), 0\rangle$. Then curl $\vec{F} = \langle 0, 0, Q_y - P_x\rangle$. Thus, Green's theorem reads

$$\oint_{\partial D} \vec{F}\cdot d\vec{r} = \iint_D (\text{curl}\,\vec{F})\cdot\hat{k}\,dA$$

Recall $\oint_{\partial D} \vec{F}\cdot d\vec{r} = \oint_{\partial D} (\vec{F}\cdot\vec{T})\,ds$, so, this is integrating the tangential component of $\vec{F}$. What of the normal component? The unit <u>outward</u> normal will be $\vec{n}(t) = \frac{1}{\|\vec{r}\,'(t)\|}\langle y'(t), -x'(t), 0\rangle$

$(\vec{r}(t) = \langle x(t), y(t), 0\rangle,\ a \le t \le b)$

Then,

$$\oint_C (\vec{F}\cdot\vec{n})\,ds = \int_a^b (\vec{F}\cdot\vec{n})(t)\,\|\vec{r}\,'(t)\|\,dt$$

$$= \int_a^b \langle P(\vec{r}(t)), Q(\vec{r}(t))\rangle \cdot \langle y'(t), -x'(t)\rangle\,dt$$

$$= \int_a^b \left(P(\vec{r}(t))\,y'(t) - Q(\vec{r}(t))\,x'(t)\right)dt$$

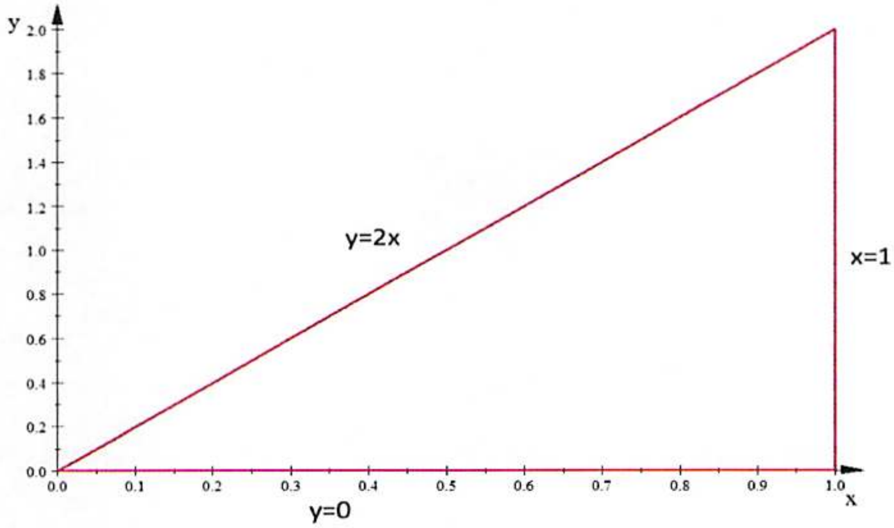
$$= \oint_C P\,dy - Q\,dx = \iint_D (P_x - (-Q_y))\,dA$$

$$= \iint_D (\mathrm{div}\,\vec{F})\,dA$$

Math 20550 - Summer 2014
Green's Theorem Worksheet

Question 1. *Compute* $\oint_C xy\,dx + x^2y^3\,dy$ *where* C *is the triangle with vertices* $(0,0)$, $(1,0)$, *and* $(1,2)$.

Answer. *The curve* C *is*



So, if C *bounds the region* D, *then*

$$\begin{aligned}\oint_C xy\,dx + x^2y^3\,dy &= \iint_D (2xy^3 - x)dA \\ &= \int_0^1 \int_0^{2x} (2xy^3 - x)dydx = \int_0^1 \left(\frac{1}{2}xy^4 - xy\right)\Big|_0^{2x} dx \\ &= \int_0^1 (8x^5 - 2x^2)dx = \left(\frac{4}{3}x^6 - \frac{2}{3}x^3\right)\Big|_0^1 = \frac{2}{3}\end{aligned}$$

Question 2. *Compute the area inside the ellipse* $x^2 + 2y^2 = 1$.

Answer. *Let E be the area inside the ellipse. Green's theorem allows us to trade the integral over the ellipse to one around the boundary of the ellipse. Here, we will use*

$$Area(E) = \iint_E dA = \frac{1}{2}\oint_{\partial E} x\,dy - y\,dx$$

First, we parametrize the boundary of E:

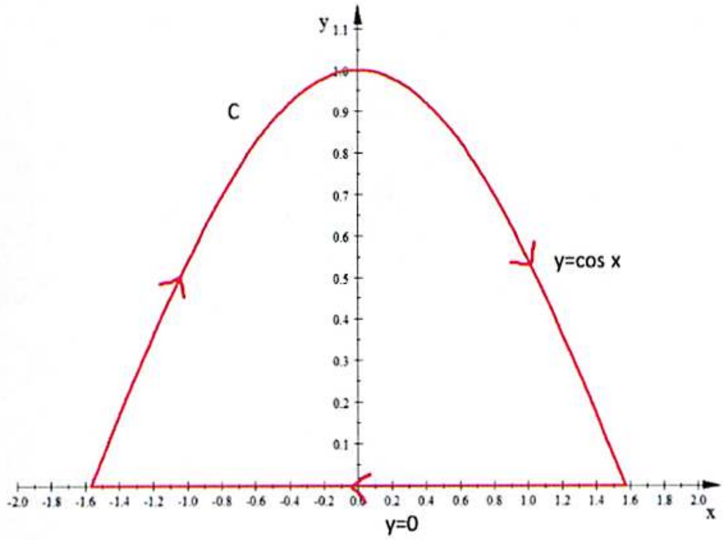
$$\mathbf{r}(t) = \left\langle \cos t, \frac{1}{\sqrt{2}}\sin t \right\rangle, 0 \le t \le 2\pi.$$

This parametrization gives it positive (counterclockwise) orientation. So,

$$\begin{aligned} Area(E) &= \frac{1}{2}\oint_{\partial E} x\,dy - y\,dx \\ &= \frac{1}{2}\int_0^{2\pi} \left((\cos t)d\left(\frac{1}{\sqrt{2}}\sin t\right) - \left(\frac{1}{\sqrt{2}}\sin t\right) d(\cos t)\right) \\ &= \frac{1}{2\sqrt{2}}\int_0^{2\pi} \left(\cos^2 t + \sin^2 t\right) dt \\ &= \frac{1}{2\sqrt{2}}\int_0^{2\pi} dt \\ &= \frac{\pi}{\sqrt{2}} \end{aligned}$$

Question 3. *Compute the line integral* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where* $\mathbf{F} = \langle e^{-x}+y^2, e^{-y}+x^2\rangle$ *and* C *consists of the arc of the curve* $y=\cos x$ *from* $\left(-\frac{\pi}{2},0\right)$ *to* $\left(\frac{\pi}{2},0\right)$ *and the line segment from* $\left(\frac{\pi}{2},0\right)$ *to* $\left(-\frac{\pi}{2},0\right)$.
$\left(\int x\cos x\,dx = x\sin x+\cos x + C\right)$

Answer. *Here is* C*, with the given orientation:*



Notice that C *has the clockwise orientation here, and so it is going the wrong way. This creates a minus sign in Green's theorem. If we call the bounded region* D*, we have*

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C (e^{-x}+y^2)dx + (e^{-y}+x^2)dy \\
&= -\iint_D (2x-2y)dA = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\cos x}(2x-2y)dydx \\
&= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2xy-y^2)\Big|_0^{\cos x}dx = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(2x\cos x - \cos^2 x\right)dx \\
&= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(2x\cos x - \frac{1}{2}-\frac{1}{2}\cos 2x\right)dx = -\left(2x\sin x + 2\cos x - \frac{x}{2}-\frac{1}{4}\sin 2x\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= -\left[\left(\pi+0-\frac{\pi}{4}\right)-\left(\pi+0+\frac{\pi}{4}\right)\right] \\
&= \frac{\pi}{2}
\end{aligned}$$

Question 4. *Compute* $\oint_C (1-y^3)dx + (x^3+e^{y^2})dy$ *where* C *is the boundary of the region between the circles* $x^2+y^2=4$ *and* $x^2+y^2=9$.

Answer. *The region is the one in between the circles of radii* 2 *and* 3, *respectively. Let* R *denote this annulus. Thus, by Green's theorem we have*

$$\begin{aligned}
\oint_C (1-y^3)dx + (x^3+e^{y^2})dy &= \iint_R (3x^2-(3y^2))dA = 3\iint_R (x^2+y^2)dA \\
&\overset{polar}{=} 3\int_0^{2\pi}\int_2^3 r^3 dr d\theta \\
&= 3\int_0^{2\pi} \frac{1}{4}r^4\Big|_2^3 d\theta \\
&= 3\int_0^{2\pi}\left(\frac{81}{4}-4\right)d\theta = 3\int_0^{2\pi}\frac{65}{4}d\theta \\
&= \frac{195\pi}{2}
\end{aligned}$$

Question 5. *Show that* $\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ *where* $\mathbf{F} = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$ *and* C *is the unit circle* $x^2 + y^2 = 1$.

Answer. *Example 5 in Section 16.4 of the textbook.*